STUDENT'S *t*-DISTRIBUTION AND CONFIDENCE INTERVALS OF THE MEAN

Suppose that we have a sample of *n* measured values $x_1, x_2, x_3, \ldots, x_n$ of a single unknown quantity. Assuming that the measurements are drawn from a normal distribution having mean μ and variance σ^2 it is reasonable to estimate these population parameters (μ, σ) with the arithmetic mean

$$\bar{x} = \frac{\sum_{k=1}^{n} x_k}{n} \tag{1}$$

and the sample variance

$$s^2 = \frac{\sum_{k=1}^{n} \left(\bar{x} - x_k\right)^2}{n-1} \tag{2}$$

where standard deviations s, σ (sample and population respectively) are positive square-roots of variances s^2, σ^2.

<u>If the sample size *n* is large</u> $(n \geq 30)$ we may estimate *confidence intervals* for the mean μ using the statistic *z,* where

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \sqrt{n}\left(\frac{\bar{x} - \mu}{\sigma}\right) \approx \sqrt{n}\left(\frac{\bar{x} - \mu}{s}\right) \tag{3}$$

and *z* is assumed to be a value of the standard normal variable *Z.*

Here we are using a theorem of statistical sampling theory:

> If all possible random samples $X_1, X_2, \ldots, X_n$ of size *n* are drawn with replacement from a normal population with mean μ and standard deviation σ, then the sampling distribution of the mean $\bar{X}$ will be approximately normally distributed with mean $\mu_{\bar{X}} = \mu$ and standard deviation $\sigma_{\bar{X}} = \sigma/\sqrt{n}$.
>
> It follows that $Z = \dfrac{\bar{X} - \mu}{\sigma_{\bar{X}}} = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \sqrt{n}\left(\dfrac{\bar{X} - \mu}{\sigma}\right)$ has a standard normal distribution.

Note that in equation (3) σ (which may be unknown) can be replaced by *s;* which is a reasonable assumption for $n \geq 30$.

Confidence intervals (CI) can be determined from the following probability statement

$$P\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) = 1 - \alpha \tag{4}$$

where $P(\ )$ is probability, α is a significance level and $-z_{\alpha/2}$, $z_{\alpha/2}$ are two tail-end values of the standard normal distribution. For a 95% CI $\alpha = 0.05$, $(1-\alpha) = 0.95$, $\alpha/2 = 0.025$ and from tables of the standard normal distribution $z_{\alpha/2} = 1.96$.

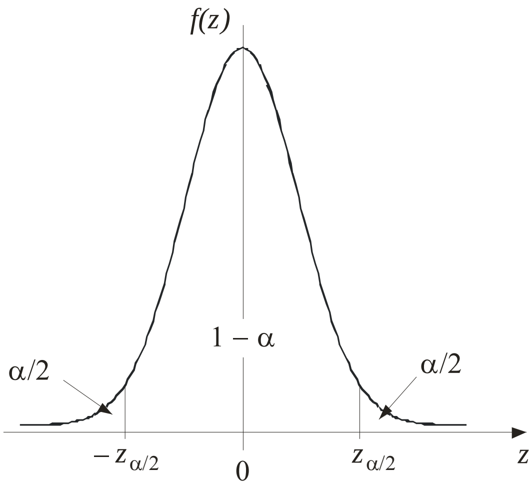


Figure 1: Standard normal distribution curve

Substituting the statistic z (equation (3)) for the standard normal variable Z in equation (4) gives

$$P\left(-z_{\alpha/2} < \sqrt{n}\left(\frac{\bar{x}-\mu}{\sigma}\right) < z_{\alpha/2}\right) = 1-\alpha \tag{5}$$

and the inequality on the left-hand-side may be re-arranged to give

$$P\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha \tag{6}$$

Thus the lower and upper confidence limits the mean μ are:

$$\begin{aligned} \text{lower confidence limit } &= \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \\ \text{upper confidence limit } &= \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \end{aligned} \tag{7}$$

For a 95% CI these limits are $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

Note that in equations (5), (6) and (7) the population standard deviation σ can be replaced by the sample estimate s. This is a reasonable assumption for $n \geq 30$.

<u>If the sample size *n* is small</u> $(n < 30)$ we may estimate *confidence intervals* for the mean μ using the statistic *t* where

$$t = \frac{\overline{x} - \mu}{s/\sqrt{n}} = \sqrt{n}\left(\frac{\overline{x} - \mu}{s}\right) \tag{8}$$

and *t* is assumed to be a value of the random variable *T* having a Student *t* distribution with $\nu = n - 1$ degrees of freedom.

The *t* distribution was introduced by W.S. Gosset who published under the pseudonym 'Student' (1908). Student's *t* distribution (or just the *t* distribution) is the distribution of the random variable $T = (\overline{x} - \mu)\big/\left(s/\sqrt{n}\right)$ of small samples whose variances s^2 may fluctuate considerably. The *t* distribution is the familiar bell-shaped curve and it can be shown that the normal distribution is the limiting case of the *t* distribution.

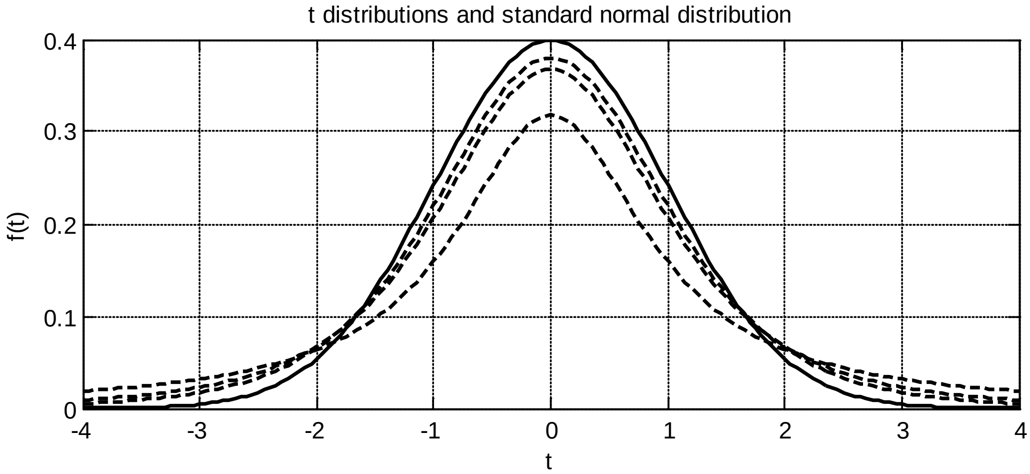


Figure 2: Standard normal distribution curve (black) and *t* distribution curves (broken) for $\nu = 1, 3, 5$ degrees of freedom. As ν increases the *t* distribution approaches the normal distribution.

Confidence intervals (CI) can be determined from the following probability statement

$$P\left(-t_{\nu,\alpha/2} < T < t_{\nu,\alpha/2}\right) = 1 - \alpha \tag{9}$$

where $P(\ )$ is probability, α is a significance level and $-t_{\nu,\alpha/2}$, $t_{\nu,\alpha/2}$ are two tail-end values of the *t* distribution with ν degrees of freedom.

For a 95% CI with $\nu = 3$ degrees of freedom, $\alpha = 0.05$, $(1-\alpha) = 0.95$, $\alpha/2 = 0.025$ and from tables of the *t* distribution for $\nu = 3$ degrees of freedom, $t_{\nu,\alpha/2} = 3.1824$.

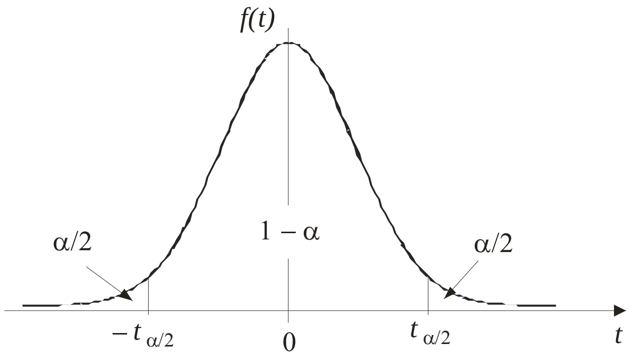


Figure 3: t distribution curve with ν degrees of freedom.

Substituting the statistic t in equation (8) for the random variable T in equation (9) gives

$$P\left(-t_{\nu,\alpha/2} < \sqrt{n}\left(\frac{\bar{x}-\mu}{s}\right) < t_{\nu,\alpha/2}\right) = 1-\alpha$$

and the inequality on the left-hand-side may be re-arranged to give

$$P\left(\bar{x} - t_{\nu,\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\nu,\alpha/2}\frac{s}{\sqrt{n}}\right) = 1-\alpha \tag{10}$$

Thus the lower and upper confidence limits the mean μ are:

$$\begin{aligned} \text{lower confidence limit } &= \bar{x} - t_{\nu,\alpha/2}\frac{s}{\sqrt{n}} \\ \text{upper confidence limit } &= \bar{x} + t_{\nu,\alpha/2}\frac{s}{\sqrt{n}} \end{aligned} \tag{11}$$

For a 95% CI and $\nu = 3$ degrees of freedom these limits are $\bar{x} \pm 3.1824\frac{s}{\sqrt{n}}$

Example 1

A distance is measured four times giving the following values. What are the 95% confidence limits of the mean? [This example is from Lauf (1983, §4, p.56]

[1] Values; residuals $v_k = (\bar{x} - x_k)$; estimate of standard deviation *s;* and degrees of freedom ν

x_k (m)	$v_k = (\bar{x} - x_k)$ (mm)	$(v_k)^2$
946.513	-27	729
.474	12	144
.450	36	1296
.507	-21	441
$\bar{x} = 946.4860$	$\sum v_k = 0$	$\sum (v_k)^2 = 2610$

estimate of variance: $s^2 = \dfrac{\sum_{k=1}^{n}\left(\bar{x} - x_k\right)^2}{n-1} = \dfrac{\sum_{k=1}^{n}\left(v_k\right)^2}{n-1}$

estimate of standard deviation: $s = \sqrt{\dfrac{\sum_{k=1}^{n}\left(v_k\right)^2}{n-1}} = \sqrt{\dfrac{2610}{4-1}} = 29.4958 \text{ mm} = 0.029 \text{ m}$

degrees of freedom: $\nu = n - 1 = 4 - 1 = 3$

[2] For a 95% CI with $\nu = 3$ degrees of freedom, $\alpha = 0.05$, $\left(1-\alpha\right) = 0.95$, $\alpha/2 = 0.025$ and from tables of the *t* distribution, $t_{\nu,\alpha/2} = 3.1824$.

[3] Confidence limits for the mean: $\bar{x} \pm t_{\nu,\alpha/2}\dfrac{s}{\sqrt{n}} = \bar{x} \pm 3.1824\left(\dfrac{0.029}{\sqrt{4}}\right) = \bar{x} \pm 0.046 \text{ m}$

and $946.440 < \mu < 946.532$

If we had used the normal distribution instead of the *t* distribution the value of $z_{\alpha/2}$ would have been 1.9600 which would have given a 95% confidence limit of $\pm 0.029 \text{ m}$, which is only 63% of the correct value (Lauf, 1983, p.56)

Example 2

The bearing and distance of a line is determined indirectly by a group of surveying students whose results are tabulated below. Discard the blunders; calculate the mean, standard deviation and range of the remaining bearings and distances and the 95% confidence limits of the mean result for the group.

126° 43'	55"	113.398		126° 44'	50"	113.393
43'	40"	.386		44'	18"	.404
40'	55"	.404		44'	11"	.398
49'	28"	.391		43'	43"	.395
43'	46"	.385		43'	53"	.404
43'	46"	.396		43'	38"	.404
43'	43"	.391		43'	55"	.392
43'	28"	.398		43'	46"	.386
43'	56"	.392		43'	39"	.372
44'	11"	.397		44'	29"	.399
				44'	05"	.389

[1] Blunders, means, standard deviations, ranges and stem & leaf plots

Ordered data

126°	49′	28″	×	113.404	✓
126°	44′	50″	×	.404	✓
126°	44′	29″	✓	.404	✓
126°	44′	18″	✓	.404	✓
126°	44′	11″	✓	.399	✓
126°	44′	11″	✓	.398	✓
126°	44′	05″	✓	.398	✓
126°	43′	56″	✓	.398	✓
126°	43′	55″	✓	.397	✓
126°	43′	55″	✓	.396	✓
126°	43′	53″	✓	.395	✓
126°	43′	46″	✓	.393	✓
126°	43′	46″	✓	.392	✓
126°	43′	46″	✓	.392	✓
126°	43′	43″	✓	.391	✓
126°	43′	43″	✓	.391	✓
126°	43′	40″	✓	.389	✓
126°	43′	39″	✓	.386	✓
126°	43′	38″	✓	.386	✓
126°	43′	28″	✓	.385	✓
126°	40′	55″	×	.372	✓

Stem & Leaf Plot plots

126° 44′ 2	9
126° 44′ 1	118
126° 44′ 0	5
126° 43′ 5	3556
126° 43′ 4	033666
126° 43′ 3	89
126° 43′ 2	8

n = 18
mean 126° 43′ 53″
st.dev. 16″
range 61″

113.40	4444
113.39	112235678889
113.38	5669
113.37	2

n = 21
mean 113.394 m
st.dev. 0.008 m
range 0.032 m

[2] For a 95% CI: $\alpha = 0.05$; $(1-\alpha) = 0.95$; $\alpha/2 = 0.025$.

For bearings; $\nu = n-1 = 17$ degrees of freedom and from tables of the *t* distribution, $t_{\nu,\alpha/2} = 2.1098$.

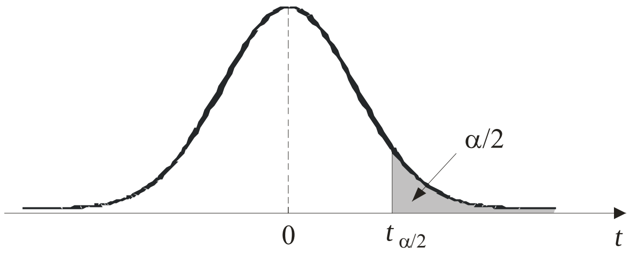
For distances; $\nu = n-1 = 20$ degrees of freedom and from tables of the *t* distribution, $t_{\nu,\alpha/2} = 2.0860$.

[3] Confidence limits for the mean: $\bar{x} \pm t_{\nu,\alpha/2} \dfrac{s}{\sqrt{n}}$

Limits for bearings: $\bar{x} \pm 2.1098\left(\dfrac{61}{\sqrt{18}}\right) = \bar{x} \pm 30.33''$ and $146^\circ\, 43'\, 20'' < \mu < 146^\circ\, 44'\, 23''$

Limits for distances: $\bar{x} \pm 2.0860\left(\dfrac{0.008}{\sqrt{21}}\right) = \bar{x} \pm 0.004\text{ m}$ and $113.390 < \mu < 113.398\text{ m}$

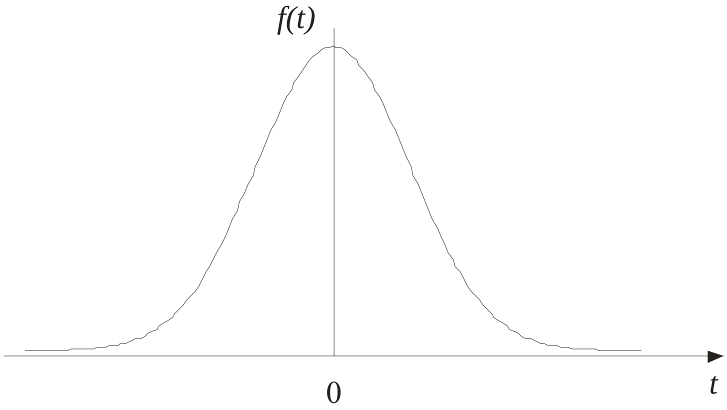
TABLE 1: t DISTRIBUTION
Probability for a given degree of freedom
(the area in the right-hand tail of the distribution)



ν	$\alpha/2$				
	0.01	0.05	0.025	0.01	0.005
1	3.0777	6.3138	12.7062	31.8205	63.6567
2	1.8856	2.9200	4.3027	6.9646	9.9248
3	1.6377	2.3534	3.1824	4.5407	5.8409
4	1.5332	2.1318	2.7764	3.7469	4.6041
5	1.4759	2.0150	2.5706	3.3649	4.0321
6	1.4398	1.9432	2.4469	3.1427	3.7074
7	1.4149	1.8946	2.3646	2.9980	3.4995
8	1.3968	1.8595	2.3060	2.8965	3.3554
9	1.3830	1.8331	2.2622	2.8214	3.2498
10	1.3722	1.8125	2.2281	2.7638	3.1693
11	1.3634	1.7959	2.2010	2.7181	3.1058
12	1.3562	1.7823	2.1788	2.6810	3.0545
13	1.3502	1.7709	2.1604	2.6503	3.0123
14	1.3450	1.7613	2.1448	2.6245	2.9768
15	1.3406	1.7531	2.1314	2.6025	2.9467
16	1.3368	1.7459	2.1199	2.5835	2.9208
17	1.3334	1.7396	2.1098	2.5669	2.8982
18	1.3304	1.7341	2.1009	2.5524	2.8784
19	1.3277	1.7291	2.0930	2.5395	2.8609
20	1.3253	1.7247	2.0860	2.5280	2.8453
21	1.3232	1.7207	2.0796	2.5176	2.8314
22	1.3212	1.7171	2.0739	2.5083	2.8188
23	1.3195	1.7139	2.0687	2.4999	2.8073
24	1.3178	1.7109	2.0639	2.4922	2.7969
25	1.3163	1.7081	2.0595	2.4851	2.7874
26	1.3150	1.7056	2.0555	2.4786	2.7787
27	1.3137	1.7033	2.0518	2.4727	2.7707
28	1.3125	1.7011	2.0484	2.4671	2.7633
29	1.3114	1.6991	2.0452	2.4620	2.7564
30	1.3104	1.6973	2.0423	2.4573	2.7500
60	1.2958	1.6706	2.0003	2.3901	2.6603
120	1.2886	1.6577	1.9799	2.3578	2.6174
∞	1.2816	1.6449	1.9600	2.3263	2.5758

For a 95% CI with $\nu = 3$ degrees of freedom, $\alpha = 0.05$, $(1-\alpha) = 0.95$, $\alpha/2 = 0.025$ and from above, $t_{\nu,\alpha/2} = 3.1824$.

Equation of the *t* distribution



The distribution of *t* (or the probability density function – pdf) is

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\left(1+\frac{t^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} \qquad (12)$$

where $\Gamma(\ )$ is the *gamma* function. The gamma function was introduced to extend the factorial function from integers to real numbers.

The factorial function for the positive integer *n* is defined by

$$n! = 1\times 2\times 3\times\cdots\times(n-2)\times(n-1)\times n$$

with *zero factorial* defined as $0! = 1$

The gamma function is defined by

$$\Gamma(\nu+1) = \int_0^\infty x^\nu e^{-x}\,dx \qquad \text{for } \nu > -1 \qquad (13)$$

with special results

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\ , \qquad \Gamma(1) = 1 \qquad (14)$$

The gamma function can be evaluated from the recurrence relationship

$$\Gamma(\nu+1) = \nu\,\Gamma(\nu) \qquad (15)$$

noting that if ν is any positive integer *n* then $\Gamma(n+1) = n!$

For example if $\nu = 9$ then in equation (12) we have (Lauf, 1983 p.55)

$$\Gamma\left(\frac{\nu+1}{2}\right) = \Gamma\left(\frac{9+1}{2}\right) = \Gamma(5) = 4! = 1\times 2\times 3\times 4 = 24$$

$$\Gamma\left(\frac{\nu}{2}\right) = \Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi} = 11.631728397$$

Hence

$$f(t) = \frac{1\times 2\times 3\times 4}{\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}\sqrt{9\pi}}\left(1+\frac{t^2}{9}\right)^{-5} = \frac{384}{315\pi}\left(1+\frac{t^2}{9}\right)^{-5}$$

If $t = 0.5$ then $f(t) = 0.3384$

Following Lauf (1983, p.56) we can see that whether is positive or negative the value of $f(t)$ is the same and so the distribution curve (the pdf) is symmetrical about the axis $t = 0$.

We may show that for large values of ν,

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \to \frac{\sqrt{2\nu}}{2} \quad \text{and} \quad \left(1+\frac{t^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} \to e^{-\frac{1}{2}t^2}$$

For example for $\nu = 30$

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} = \frac{\Gamma\left(\frac{31}{2}\right)}{\Gamma(15)} = \frac{334838609873.555}{87178291200} = 3.8408 \text{ and } \frac{\sqrt{2\nu}}{2} = \frac{\sqrt{60}}{2} = 3.8729$$

and for $\nu = 30$ and $t = 0.5$

$$\left(1+\frac{t^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} = \left(1+\frac{(0.5)^2}{30}\right)^{-\frac{31}{2}} = 0.8793 \text{ and } e^{-\frac{1}{2}t^2} = e^{-0.0125} = 0.8825$$

So that $f(t) \to \frac{\sqrt{2\nu}}{2\sqrt{\pi\nu}} e^{-\frac{1}{2}t^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$ which is the normal distribution.

This means that we may always use the Student *t* distribution whether ν is small or large (noting that the normal distribution is valid only for large values of ν)

The probability *p* that a single observation from the *t* distribution with ν degrees of freedom will fall in the interval $[-\infty, x)$ is that area under the curve between $-\infty$ and $t = x$ and

$$p = \int_{t=-\infty}^{t=x} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\left(1+\frac{t^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} dt \qquad (16)$$

For given values of *p* the corresponding value of *t* can be evaluated from equation (16) by numerical integration.

For example, for a 95% CI $\alpha = 0.05$, $(1-\alpha) = 0.95$ and $\alpha/2 = 0.025$ is the area under the right-hand tail of the distribution curve. This corresponds to a value of $t = t_{\alpha/2}$ and the area under the curve between $-\infty$ and $t = t_{\alpha/2}$ is equal to $p = 0.975$ and $t_{\alpha/2}$ obtained from equation (16)

The values for Table 1 were obtained by such methods using function *tinv( )* from Matlab's Statistics Toolbox.

```
table = zeros(30,6);
for k = 1:30
    table(k,1) = k;
    row = tinv([0.9 0.95 0.975 0.99 0.995],k);
    table(k,2:6) = row;
end
table
```

Reference

Lauf, G.B., (1983), *The Method of Least Squares with applications in surveying*, Tafe Publications Unit, Collingwood, Australia.